

# On completing three cyclic transversals to a latin square

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## Abstract

Let  $P$  be a partial latin square of prime order  $p > 7$  consisting of three cyclically generated transversals. Specifically, let  $P$  be a partial latin square of the form:

$$P = \{(i, c + i, s + i), (i, c' + i, s' + i), (i, c'' + i, s'' + i) \mid 0 \leq i < p\}$$

for some distinct  $c, c', c''$  and some distinct  $s, s', s''$ . In this paper we show that any such  $P$  completes to a latin square which is diagonally cyclic.

Keywords: latin square, diagonally cyclic latin square, transversals, complete mappings, toroidal semi-queens.

## 1 Background information

A *latin square* of order  $n$  is an  $n \times n$  array of symbols such that each cell contains one symbol and each symbol occurs once in each row and once in each column. In this paper, rows, columns and symbols are taken from the set  $N = \{0, 1, \dots, n-1\}$  and are always calculated modulo  $n$ . We use the notation  $i \circ j$  to denote the symbol in cell  $(i, j)$  of a latin square  $L = L^\circ$ . We may denote  $L$  a set of ordered (row, column, symbol) triples of the form  $(i, j, i \circ j)$ . A *partial latin square*, as its name suggests, is a partially filled-in  $n \times n$  array of symbols such that each cell contains at *most* one symbol and each symbol occurs at most once in each row and at at most once in each column.

A latin square  $L^\circ$  is said to be *diagonally cyclic* if  $n$  is odd and for each cell  $(i, j)$ ,  $i \circ j = k$  implies that  $(i+1) \circ (j+1) = k+1$ . Diagonally cyclic squares of even order do not exist; for a nice proof see [6]. Henceforth we assume that  $n$  is odd.

Let  $j$  be coprime to some odd  $n$  and  $j \neq 1$ . If we define  $\circ_j$  by  $0 \circ_j i = ij \pmod n$  for each  $i$ , a valid diagonally cyclic latin square is generated, which we denote by  $B_{n,j}$ . The following diagram shows  $B_{5,3}$ .

0	3	1	4	2
3	1	4	2	0
1	4	2	0	3
4	2	0	3	1
2	0	3	1	4

Clearly a diagonally cyclic latin square is defined by the ordering of symbols in row 0. However, not all orderings of the first row will complete to a diagonally cyclic latin square, as symbols may not be repeated within a column. In fact, row 0 “generates” a diagonally cyclic latin square if and only if the operation  $f(i) = 0 \circ i - i$  is a permutation of the set  $N$ .

Let  $L$  be a diagonally cyclic latin square of order  $n$ . For any constant integer  $c$ , we define  $L \oplus_1 c$  to be the diagonally cyclic latin square formed by cycling the rows of  $L$  by  $c$  (modulo  $n$ ). That is,

$$L \oplus_1 c = \{(i+c, j, k) \mid (i, j, k) \in L\}.$$

Clearly such a transformation is invertible, and thus gives rise to equivalence classes of diagonally cyclic latin squares. We thus often assume, without loss of generality, that  $0 \circ 0 = 0$ . In fact, if  $0 \circ 0 = 0$  we say that the diagonally cyclic latin square is in *standard order*. Similarly we define  $L \oplus_2 c$  and  $L \oplus_3 c$  to be the diagonally latin squares formed by adding constant  $c$  to each column or symbol, respectively.

Let  $L$  be a diagonally cyclic latin square in standard order. For any  $c$  coprime to  $n$ , we define  $L \times c$  to be the diagonally cyclic latin square generated by operation  $\circ_c$ , where  $i \circ_c j = (ic^{-1} \circ jc^{-1})c \pmod{n}$ , for each  $i, j$  such that  $0 \leq i, j \leq n-1$ . (Informally, the first row of  $L \times c$  is formed by multiplying each column and symbol of the first row of  $L$  by  $c$ .) Again, such a transformation is invertible. For the curious reader, other equivalences of diagonally cyclic latin squares are given in Lemma 2.1 of [5].

So suppose that  $L$  is a diagonally cyclic latin square of prime order  $p$  containing the partial latin square  $P$  defined as in the abstract. Consider the isomorphic latin square

$$L' = (((L \oplus_1 (-s)) \oplus_3 (-c)) \times ((c' - c)^{-1})).$$

Observe that  $L'$  is in standard order, with symbol  $(s' - s)/(c' - c)$  in row 0 and column 1.

In Section 4, when we prove the main result, we thus assume, without any loss of generality, that  $0 \circ 0 = 0$  and  $0 \circ 1 = j$ .

In any diagonally cyclic latin square  $L$ , the set of symbols  $L(\alpha) = \{(i, \alpha + i, 0 \circ \alpha + i) \mid 0 \leq i \leq n-1\}$  contains each row, each column and each symbol exactly once and is thus a *transversal*. We denote  $L(\alpha)$  as a *cyclic transversal* of  $L$ . In fact, the cyclic transversals  $L(\alpha)$ ,  $0 \leq \alpha \leq n-1$  are pairwise disjoint. Thus any diagonally cyclic latin square is orthogonal to the latin square defined by the relation  $i \circ j = j - i$ .

The research in this paper is motivated by the following problem posed by Alspach and Heinrich [1]: For each  $k$ , does there exist an  $N(k)$  such that if  $k$  transversals of a partial latin square of order  $n > N(k)$  are prescribed, the square can always be completed? The existence of idempotent latin squares for every order  $n \neq 2$  shows that  $N(1) = 3$ . Häggkvist and Daykin [3] have shown that every partial  $n \times n$  latin square where each row, column and symbol is used at most  $2^{-9}\sqrt{n}$  times is completable whenever  $n$  is divisible by 16. However it is as yet unconfirmed that  $N(2)$  exists. Grützmüller [6] showed that  $N(k) \geq 4k - 1$ . Computational results on latin squares of small order support the conjecture that  $N(k) = 4k - 1$ .

As the above problem is difficult, it seems sensible to consider the following modified version, proposed by Grützmüller [5]: Does there exist an odd constant  $C(k)$  such that if  $k$  *cyclically generated* diagonals of a partial latin square of odd order  $n \geq C(k)$  are prescribed, the square can always be completed? Grützmüller showed that  $C(2) = 3$  ([6]) and that  $C(k) \geq 3k - 1$  for  $k \geq 3$  ([5]).

In this paper we provide some evidence that  $C(3)$  may be equal to 9. Specifically, we show (in Theorem 7) that if  $P$  is a partial latin square of prime order  $p > 7$  of the form

$$P = \{(i, c + i, s + i), (i, c' + i, s' + i), (i, c'' + i, s'' + i) \mid 0 \leq i < p\}$$

for some distinct  $c, c', c''$  and some distinct  $s, s', s''$ , then  $P$  has a completion to a (diagonally cyclic) latin square.

In our proof, in Section 2 we first identify a method to reorder certain cyclically generated transversals within  $B_{p,j}$  to form a new diagonally cyclic latin square. This reordering or *trade* is algebraically defined and the transversals are based on some linear transformation of the quadratic residues mod  $p$ . We are thus able to redefine the problem above in terms of simultaneous equations, where instead of a precise solution to each equation, we instead require some information about which coset the solution belongs to. Here cosets are taken from the multiplicative group of the field of prime order  $p$ .

When  $p$  is large enough (specifically, when  $p \geq 191$ ) in Section 4 we conveniently exploit Weil's theorem to obtain the required solution. Computational methods are then employed in Section 5 for the remaining, smaller primes.

In the next section we describe some combinatorial objects which are equivalent to diagonally cyclic latin squares.

## 2 Combinatorial equivalences

Diagonally cyclic latin squares have a number of intriguing equivalences. Firstly, diagonally cyclic latin squares of order  $n$  are equivalent to transversals in the latin square  $B_n$ , which is precisely the operation table for the integers modulo  $n$ . To see this, let  $L$  be a diagonally cyclic latin square with operation  $\circ$ . Let  $P \subseteq B_n$  be the partial latin square defined by  $P = \{(0 \circ i - i, i, 0 \circ i) \mid 0 \leq i \leq n - 1\}$ . Then  $P$  is a transversal. Conversely,

let  $P$  be a transversal of  $B_n$ . Then, for each triple  $(r, c, r + c \pmod n) \in P$ , define  $0 \circ c = r + c$ . Then  $\circ$  generates a valid diagonally cyclic latin square.

Another equivalence involves placing  $n$  semi-queens on a toroidal  $n \times n$  chessboard such that no two semi-queens may attack each other. A semi-queen attacks any piece in its row and column, but only on the *ascending* diagonals; i.e. those which begin in the lower left and finish in the upper right. On a toroidal chessboard these diagonals “wrap around” in the obvious fashion. Thus if we replace the chessboard with the operation table for the integers modulo  $n$ , we see that each queen must lie on a different symbol, and no queens may share a common row or a common column. So a set of  $n$  semi-queens which may not attack each other on the next turn is equivalent to a transversal within  $B_n$ . For more detail on the semi-queen problem, we refer the reader to [9].

Diagonally cyclic latin squares in standard order are also equivalent to *complete mappings* of the cyclic group. We refer the reader to [7] for more detail on complete mappings.

### 3 How to reorder cyclic transversals within $B_{n,j}$ via number theory

Consider the diagonally cyclic latin square  $B_{11,6}$  below. If we replace each symbol in bold with its subscript, we obtain another diagonally cyclic latin square. In fact, the bold symbols in the first row are precisely the quadratic residues modulo 11, and the subscripts in the first row are obtained by multiplication by 4. This example is part of a general construction given in the following theorem.

**Example 1.**

0	6	1 <sub>4</sub>	7	2	8	3 <sub>1</sub>	9 <sub>3</sub>	4 <sub>5</sub>	10	5 <sub>9</sub>
6 <sub>10</sub>	1	7	2 <sub>5</sub>	8	3	9	4 <sub>2</sub>	10 <sub>4</sub>	5 <sub>6</sub>	0
1	7 <sub>0</sub>	2	8	3 <sub>6</sub>	9	4	10	5 <sub>3</sub>	0 <sub>5</sub>	6 <sub>7</sub>
7 <sub>8</sub>	2	8 <sub>1</sub>	3	9	4 <sub>7</sub>	10	5	0	6 <sub>4</sub>	1 <sub>6</sub>
2 <sub>7</sub>	8 <sub>9</sub>	3	9 <sub>2</sub>	4	10	5 <sub>8</sub>	0	6	1	7 <sub>5</sub>
8 <sub>6</sub>	3 <sub>8</sub>	9 <sub>10</sub>	4	10 <sub>3</sub>	5	0	6 <sub>9</sub>	1	7	2
3	9 <sub>7</sub>	4 <sub>9</sub>	10 <sub>0</sub>	5	0 <sub>4</sub>	6	1	7 <sub>10</sub>	2	8
9	4	10 <sub>8</sub>	5 <sub>10</sub>	0 <sub>1</sub>	6	1 <sub>5</sub>	7	2	8 <sub>0</sub>	3
4	10	5	0 <sub>9</sub>	6 <sub>0</sub>	1 <sub>2</sub>	7	2 <sub>6</sub>	8	3	9 <sub>1</sub>
10 <sub>2</sub>	5	0	6	1 <sub>10</sub>	7 <sub>1</sub>	2 <sub>3</sub>	8	3 <sub>7</sub>	9	4
5	0 <sub>3</sub>	6	1	7	2 <sub>0</sub>	8 <sub>2</sub>	3 <sub>4</sub>	9	4 <sub>8</sub>	10

**Theorem 2.** Let  $(a, b, c)$  be a solution to the equation:

$$(1 - j)a^m + jb^m \equiv c^m \pmod{p} \quad (1)$$

where  $p$  is a prime,  $j \neq 1$ ,  $j$  is coprime to  $p$ ,  $m \geq 2$ ,  $m$  divides  $p - 1$  and  $a^m$ ,  $b^m$  and  $c^m$  are non-zero and pairwise distinct  $(\text{mod } p)$ . Let  $\alpha \neq 0$  and  $\gamma$  be constants modulo  $p$ . Next, replace each symbol in the first row of  $B_{p,j}$  of the form

$$\alpha\beta^m + \gamma$$

(where  $\beta \neq 0$ ) with the symbol

$$\alpha(b/c)^m\beta^m + \gamma.$$

Then this new first row generates a valid diagonally cyclic latin square.

*Proof.* Let  $L$  be the square array (at this stage we haven't proved that it is latin) generated by the new first row. Since  $b^m \neq c^m$ ,  $(b/c)^m$  acts as a derangement on the set of  $m$ th powers, so the symbols in the first row of  $L$  (and thus in all rows) are distinct.

It suffices, then, to check that all symbols in the first column are distinct. Within  $B_{p,j}$ , symbol  $\alpha\beta^m + \gamma$  occurs in column  $j^{-1}(\alpha\beta^m + \gamma)$  in row 0. So this cyclic transversal contains the symbol  $(\alpha\beta^m + \gamma)(1 - j^{-1})$  in column 0. Within  $L$  this is replaced by

$$\begin{aligned} (\alpha(b/c)^m\beta^m + \gamma) - j^{-1}(\alpha\beta^m + \gamma) &= \alpha\beta^m((b/c)^m - j^{-1}) + \gamma(1 - j^{-1}) \\ &= (\alpha\beta^m(a/c)^m + \gamma)(1 - j^{-1}). \end{aligned}$$

Again, as  $(a/c)^m$  permutes the  $m$ th powers, the symbols in the first column (and thus in all columns) are distinct.  $\square$

Observe that  $7^2 + 1^2 = 2 \times 5^2$ . Thus, for  $m = 2$ , the triple  $(a, b, c) = (7, 1, 5)$  is a valid solution of Equation (1) for  $j = (p + 1)/2$  and any prime  $p > 7$ . The earlier Example 1 demonstrates this for  $p = 11$ ,  $\alpha = 1$  and  $\gamma = 0$ .

## 4 Completing sets of cyclic transversals

We now focus on the problem of completing three arbitrary cyclic transversals to a (diagonally cyclic) latin square. In this section, we restrict ourselves to the case where  $p$  is a prime. We denote the finite field of size  $p$  by  $\text{GF}(p)$ .

We may assume, without loss of generality, that we have transversals generated by 0 and  $j$  in cells  $(0, 0)$  and  $(0, 1)$ , respectively. We must have  $j \neq 1$  for this to be a valid partial latin square. Our third transversal is arbitrary; assume that it is generated by symbol  $e$  in column  $c$ . So that no symbols repeat in a row or in a column, we have as necessary conditions  $c \notin \{0, 1\}$ ,  $e \notin \{0, j, c\}$  and  $e - c \neq j - 1$ . If  $e = jc$ , then these three transversals are contained within  $B_{p,j}$  and thus complete to a latin square. So we henceforth assume also that  $e \neq jc$ .

Our aim is to apply Theorem 2 to  $B_{p,j}$  to obtain a latin square which contains the above three cyclic transversals. To transform the third transversal, it is sufficient to find  $m$ ,  $\alpha$ ,  $\beta$  and  $\gamma$ , with  $m \mid p - 1$ ,  $\alpha \neq 0$ ,  $\beta \neq 0$ , such that

$$jc = \alpha\beta^m + \gamma \tag{2}$$

$$e = \alpha x\beta^m + \gamma \tag{3}$$

where  $x = (b/c)^m$  for some appropriate solution  $(a, b, c)$  to Equation 1. Equivalently (considering the conditions of Theorem 2), we assume that  $x$  is an  $m$ th power modulo  $p$ ,  $x \notin \{0, 1\}$  and that if we define

$$F(x) = \frac{1 - jx}{1 - j},$$

then  $F(x)$  is also an  $m$ th power and  $F(x) \notin \{0, 1\}$ .

Equations (2) and (3) imply that a given  $x$  determines  $\gamma$  and the product  $\alpha\beta^m$ :

$$\gamma = \frac{xjc - e}{x - 1}, \quad (4)$$

$$\alpha\beta^m = \frac{e - jc}{x - 1}. \quad (5)$$

Since  $x \neq 1$  these equations are well-defined. Given a valid  $x$  and  $m$ , it is always possible to choose such  $\alpha$ ,  $\beta$  and  $\gamma$ . However, in this process of transformation we do not wish to alter the first two transversals in columns 0 and 1.

Equivalently,  $-\gamma/\alpha$  and  $(j - \gamma)/\alpha$  must not be  $m$ th powers modulo  $p$  (although they may be equal to 0). But Equation (5) implies that  $\alpha$  is a non-zero  $m$ th power if and only if  $(e - jc)/(x - 1)$  is a non-zero  $m$ th power. Thus, the following expressions must not be non-zero  $m$ th powers modulo  $p$ :

$$G(x) = \frac{-\gamma(x - 1)}{e - jc} = \frac{e - xjc}{e - jc}$$

$$H(x) = \frac{(j - \gamma)(x - 1)}{e - jc} = \frac{xj(1 - c) + e - j}{e - jc}$$

Note that if  $H(x) \neq 1$  then  $x \neq 1$ . In turn, if  $x \neq 1$  then  $F(x) \neq 1$ . So if we assume that  $H(x) \neq 1$  it is unnecessary to specify  $x \neq 1$  and  $F(x) \neq 1$  as conditions. In summary, we have the following.

**Lemma 3.** *Let  $p$  be prime. Let  $c$ ,  $e$  and  $j$  be residues mod  $p$  such that  $c \notin \{0, 1\}$ ,  $j \notin \{0, 1\}$  and  $e \notin \{0, j, c, c + j - 1, jc\}$ . Let  $x \neq 0$  be an  $m$ th power mod  $p$  such that  $F(x)$  is a non-zero  $m$ th power mod  $p$ , and  $G(x)$  and  $H(x)$  are not non-zero  $m$ th powers mod  $p$ . Then, the three cyclic transversals generated by  $0 \circ 0 = 0$ ,  $0 \circ 1 = j$  and  $0 \circ c = e$  can be completed to a diagonally cyclic latin square.*

If we fix a value of  $x$  there will be some inappropriate choices of  $c$ ,  $e$  and  $j$ . However, we next show that for sufficiently large primes, it is possible to find such a  $x$  for any choice of  $c$ ,  $e$  and  $j$  that satisfies the necessary conditions outlined above.

The techniques that follow are similar to those applied to constructing cyclic triplewhist tournaments in [2]. We need the following version of Weil's Theorem (see [8], Theorem 5.38). Recall that a *multiplicative character*  $\chi$



over  $\text{GF}(p)$  is a homomorphism from the multiplicative group of  $\text{GF}(p) - \{0\}$  into the multiplicative group of unitary complex numbers. When Weil's theorem is applied it is also understood that  $\chi(0) = 0$  for any multiplicative character  $\chi$ .

**Theorem 4.** (*Weil's Theorem*) Let  $\chi$  be a multiplicative character of the finite field  $\text{GF}(p)$  with order  $m > 1$  and let  $f$  be a polynomial of  $\text{GF}(p)[x]$  of degree  $d$  which is not of the form  $kg^m$  for some constant  $k \in \text{GF}(p)$  and some  $g \in \text{GF}(p)[x]$ . Then:

$$\left| \sum_{c \in \text{GF}(p)} \chi(f(c)) \right| \leq (d-1)p^{1/2}.$$

For  $m = 2$ , we will use the *quadratic character*  $\eta$  defined as follows on  $\text{GF}(p) - \{0\}$ :

$$\eta(x) = \begin{cases} 1 & \text{if } x \text{ is a quadratic residue;} \\ -1 & \text{otherwise.} \end{cases}$$

For  $m = 2$  we can obtain a more explicit version of Weil's Theorem ([8], Theorem 5.48):

**Theorem 5.** Let  $\eta$  be the quadratic character of  $\text{GF}(p)$  and let  $f = a_2x^2 + a_1x + a_0 \in \text{GF}(p)[x]$  with  $p$  odd and  $a_2 \neq 0$ . If  $a_1^2 - 4a_0a_2 \neq 0$  then:

$$\sum_{c \in \text{GF}(p)} \eta(f(c)) = -\eta(a_2).$$

We can now show the main theorem of this section:

**Theorem 6.** If  $P$  is a partial latin square of prime order  $p \geq 191$  comprising of three arbitrary transversals, then  $P$  has a completion to a (diagonally cyclic) latin square.

*Proof.* Setting  $m = 2$ , it suffices to show that the conditions of Lemma 3 hold for any valid choices of  $j$ ,  $c$ , and  $e$ . We wish to show that there exists some  $x \in \text{GF}(p)$  such that  $\eta(x) = 1$ ,  $\eta(F(x)) = 1$ ,  $(\eta(G(x)) = -1 \text{ or } 0)$  and

( $\eta(H(x)) = -1$  or  $0$ ). In fact, we show that there exists specifically such an  $x$  for which  $\eta(x) = 1$ ,  $\eta(F(x)) = 1$ ,  $\eta(G(x)) = -1$  and  $\eta(H(x)) = -1$ .

Equivalently, we will show that for  $p \geq 191$ , the following set  $A$  is non-empty:

$$A = \{x \mid \eta(x) = \eta(F(x)) = 1 \text{ and } \eta(G(x)) = \eta(H(x)) = -1\}.$$

Define

$$J(x) = [1 + \eta(x)][1 + \eta(F(x))][1 - \eta(G(x))][1 - \eta(H(x))]$$

and

$$S = \sum_{x \in \text{GF}(p)} J(x).$$

If any of  $x$ ,  $F(x)$ ,  $G(x)$  or  $H(x)$  equals 0, then  $J(x) \leq 8$ . If  $x \in A$  then  $J(x) = 16$ . For other values of  $x \in \text{GF}(p)$ ,  $J(x) = 0$ . Thus  $S \leq 16|A| + 32$ . So it suffices to show that  $S > 32$ .

By expanding  $S$  and using the fact that  $\eta$  is a homomorphism, we can express  $S$  as  $p$  plus a series of terms of the form  $\sum_{c \in \text{GF}(p)} \eta(K(x))$ , where  $K(x)$  is a product of non-repeated factors from the set  $\{x, F(x), -G(x), -H(x)\}$ .

For any (non-constant) linear function  $g(x)$ , it is well-known that

$$\sum_{x \in \text{GF}(p)} g(x) = 0.$$

From the necessary conditions on  $j$ ,  $c$  and  $e$ , each of the quadratics  $xF(x)$ ,  $xG(x)$ ,  $xH(x)$ ,  $F(x)G(x)$ ,  $F(x)H(x)$  and  $G(x)H(x)$  have no repeated linear factors and thus each has a non-zero discriminant.

It follows that the conditions of Weil's theorem hold for each of the terms of the form  $K(x)$  in the expansion of  $S$ . There are  $\binom{4}{2}$  quadratic terms, so from Theorem 5, these contribute at least  $-6$  to  $S$ . Next, there are  $\binom{4}{3}$  cubic terms, so from Theorem 4 with  $d = 3$ , these contribute at least  $-8\sqrt{p}$  to  $S$ . Finally there is one quartic term, so from Theorem 4 with  $d = 4$ , this contributes at least  $-3\sqrt{p}$  to  $S$ . Thus,  $S \geq p - 11\sqrt{p} - 6$ . But  $p - 11\sqrt{p} - 6 > 32$  for any prime  $p \geq 191$ .  $\square$

## 5 Computational results

By using computational methods for primes less than 191, Theorem 6 from the previous section can be improved to the following:

**Theorem 7.** *If  $P$  is a partial latin square of prime order  $p > 7$  of the form*

$$P = \{(i, c + i, s + i), (i, c' + i, s' + i), (i, c'' + i, s'' + i) \mid 0 \leq i < p\}$$

*for some distinct  $c, c', c''$  and some distinct  $s, s', s''$ , then  $P$  has a completion to a (diagonally cyclic) latin square.*

Note that in [5] it is shown that three cyclic transversals do not always complete to diagonally cyclic latin squares of order 7.

Two approaches were used to do the small cases computationally. Firstly, we checked all the instances where the conditions of Lemma 3 were satisfied for an appropriate  $m$ . This approach provided complete solutions for all primes between 61 and 181 (inclusive), and most solutions for primes between 11 and 59, leaving a total of 2076 exceptions (choices of  $j$ ,  $c$ , and  $e$ ) for smaller primes:

prime	11	13	17	19	23	29	31	41	47	59
#excep	180	54	306	84	672	252	66	12	300	150

The exceptions  $11 \leq p \leq 59$  were solved using a randomised depth first search algorithm [4].

## 6 Case $p = 11$

Here we present the computational data for  $p = 11$ . Each of the following four columns contain the exceptions (triples  $c, e, j$ ) and the first row of the diagonally cyclic latin square which was found by backtrack search for the given triple. The symbol ‘a’ denotes 10.

2 1 4 04175a29368	4 5 7 07a65429138	6 5 4 04a92758136	8 7 3 03a69148725
2 1 8 08146a92573	4 5 8 0869543a271	6 5 7 07361a59428	8 7 6 065a2938741
2 3 5 0538a429617	4 6 4 04a86175329	6 7 8 08591376a24	8 7 8 084692157a3
2 3 9 09358a42716	4 6 8 08376a42519	6 8 6 063a2984715	8 7 9 0938614a725
2 5 2 0258a149763	4 6 9 09376a15482	6 8 9 09615a83742	8 7 10 0a685139742
2 5 9 09582176a43	4 7 3 038271a5964	6 9 3 03148a92657	8 9 4 048a2751963
2 6 7 076194835a2	4 7 5 05397421a68	6 9 6 061a2798543	8 9 5 05417a32986
2 6 8 08625391a47	4 8 7 07328a94651	6 10 4 048136a9257	8 10 7 07185394a26
2 7 4 0472a185963	4 8 9 0942817a635	7 1 2 0296a481573	8 10 8 08721695a43
2 7 5 057a6432918	4 9 3 03a49826175	7 1 3 038a5491627	9 1 3 036a9248715
2 8 3 038279415a6	4 9 4 04a59172683	7 1 4 04928371a65	9 1 9 09642a83715
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4 5 6 06925814a73	6 5 3 0398a654172	8 6 7 07a43981652	10 9 5 0536a184279

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## References

- [1] Brian Alspach and Katherine Heinrich. Matching designs. *Australas. J. Combin.*, 2:39–55, 1990. Combinatorial mathematics and combinatorial computing, Vol. 2 (Brisbane, 1989).
- [2] Marco Buratti. Existence of  $Z$ -cyclic triplewhist tournaments for a prime number of players. *J. Combin. Theory Ser. A*, 90(2):315–325, 2000.
- [3] David E. Daykin and Roland Häggkvist. Completion of sparse partial Latin squares. In *Graph theory and combinatorics (Cambridge, 1983)*, pages 127–132. Academic Press, London, 1984.
- [4] Carla P. Gomes, Bart Selman, and Henry Kautz. Boosting combinatorial search through randomization. In *Proceedings of the Fifteenth National Conference on Artificial Intelligence (AAAI'98)*, pages 431–437, Madison, Wisconsin, 1998.
- [5] Martin Grützmüller. Completing partial Latin squares with two cyclically generated prescribed diagonals. *J. Combin. Theory Ser. A*, 103(2):349–362, 2003.
- [6] Martin Grützmüller. Completing partial Latin squares with prescribed diagonals. *Discrete Appl. Math.*, 138(1-2):89–97, 2004. Optimal discrete structures and algorithms (ODSA 2000).
- [7] Hsiang J, Shieh Y, and Chen Y. The cyclic complete mappings counting problems. In *PaPS: Problems and Problem Sets for ATP Workshop in conjunction with CADE-18 and FLoC 2002*. Copenhagen, Denmark, 2002.
- [8] Rudolf Lidl and Harald Niederreiter. *Finite fields*, volume 20 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1983. With a foreword by P. M. Cohn.
- [9] Igor Rivin, Ilan Vardi, and Paul Zimmermann. The  $n$ -queens problem. *Amer. Math. Monthly*, 101(7):629–639, 1994.